

Hamiltonian Loop Group Actions and T-Duality for group manifolds

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Abstract

We carry out a Hamiltonian analysis of Poisson-Lie T-duality based on the loop geometry of the underlying phase spaces of the dual sigma and WZW models. Duality is fully characterized by the existence of equivariant momentum maps on the phase spaces such that the reduced phase space of the WZW model and a pure central extension coadjoint orbit work as a bridge linking both the sigma models. These momentum maps are associated to Hamiltonian actions of the loop group of the Drinfeld double on both spaces and the duality transformations are explicitly constructed in terms of these actions. Compatible dynamics arise in a general collective form and the resulting Hamiltonian description encodes all known aspects of this duality and its generalizations.

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INTRODUCTION

Poisson-Lie T-duality [1] refers to a non-Abelian duality between two $1 + 1$ σ -models describing the motion of a string on targets being a dual pair of Poisson-Lie groups [2], composing a perfect Drinfeld double group [3]. The Lagrangians of the models are written in terms of the underlying bialgebra structure of the Lie groups, and Poisson-Lie T-duality stems from the self dual character the Drinfeld double. Classical T-duality transformation comes to relate some *dualizable* subspaces of these phase-spaces, mapping solutions reciprocally. It comes to generalize the Abelian $R \longleftrightarrow R^{-1}$ [4] and non-Abelian $G \longleftrightarrow \mathfrak{g}^*$ [6],[5] dualities which hold at classical and quantum level. Former version appears tied to the dual symmetric structure of the target manifolds of dual models [7]. The Poisson Lie T-duality reproduces all of them when the symplectic structure on the Drinfeld double $D = G \bowtie G^*$ [8] is reduced to the cotangent bundle T^*G with G a trivial Poisson-Lie group.

A generating functional for PLT duality transformations [1] [9] is constructed from the symplectic structure on D , and it was shown [10] from algebraic properties in the dual Lagrangians they are canonical ones (although their domains remain unclear). Also, for closed string models, a Hamiltonian description [11] reveals that there exists Poisson maps from the *T-dual* phase-spaces to the centrally extended loop algebra of the Drinfeld double, and it holds for any hamiltonian dynamics on this loop algebra and lifted to the T-dual phase-spaces.

In the pioneer works [1] [12], it was proposed a WZW-type model with target on the Drinfeld double group D from which a PL T-dual pair of σ -models are obtained, providing a common roof and making clear how PL T-duality works: solutions on a σ -model are lifted to the WZW model on D and then projected to the dual one. This setting makes natural the appearance of the symplectic structure on D , in the generating functional of the duality transformations. However, in contrast with the hamiltonian approach in [11], the dynamic of the involved models were fixed to a very particular form. It was also noted that PL T-duality just work on some subspaces satisfying some *dualizable* conditions expressed as monodromy constraints. In the hamiltonian approach to the abelian $R \leftrightarrow R^{-1}$ and non Abelian duality $G \leftrightarrow \mathfrak{g}^*$, the dualizable spaces were well characterized [5] leading, for example, to the momentum-winding exchange, but it is unclear how to do the same in PL case.

An approach in the framework of bicrossed product of Lie algebras is presented in ref. [13], constructing and classifying many dual models for the quasitriangular case, studying the possible orthogonal decomposition of the Drinfeld double algebra and fixing appropriated hamiltonian dynamics.

The main aim of this work is to carry out a unified description of classical PL T-duality based on the symplectic geometry of the loop groups spaces involved in sigma and WZW models. We encode it in the commutative diagram

$$\begin{array}{ccccc}
 & & (L\mathfrak{d}_\Gamma^*; \{, \}_\text{KK}) & & \\
 & \nearrow \mu & \uparrow \hat{\Phi} & \nwarrow \tilde{\mu} & \\
 (T^*LG; \omega_o) & & & & (T^*LG^*; \tilde{\omega}_o) \\
 & \nwarrow & \downarrow & \nearrow & \\
 & & (\Omega D; \omega_{\Omega D}) & &
 \end{array} \tag{1}$$

where the left and right vertices are the phases spaces of the σ -models, with the canonical Poisson (symplectic) structures, $L\mathfrak{d}_\Gamma^*$ is the dual of the centrally extended Lie algebra of LD with the Kirillov-Kostant Poisson structure, and ΩD is the symplectic manifold of based loops. In particular, alike $\hat{\Phi}$, we derive μ and $\tilde{\mu}$ as momentum maps associated to hamiltonian actions of the centrally extended loop group LD^\wedge on the σ -models. These actions split the tangent bundles of the preimages under μ and $\tilde{\mu}$ of the pure central extension orbit, and the dualizable subspaces are identified as the orbits of ΩD which turn to be the symplectic foliation. We shall show that the restriction of the diagram to these subspaces, with symplectic arrows, gives precise description of the PL T-duality embodying its essential features and providing a clear framework to link with other approaches. From this setting, we shall be able to build dual hamiltonian models by taking any suitable hamiltonian function on the loop algebra of the double and lifting it in a collective form [21]. For particular choices, the lagrangian formalism will be reconstructed obtaining the known dual σ -models and the master WZW-like model encoding them.

This work is organized as follows: in Section I, we review the main features of the symplectic geometry of the WZW model; in Section II, we describe the actions of the LD^\wedge on the phase spaces of the sigma model with target G and G^* , constructing the associated momentum maps and explaining the connection of the group of based loops with this phase spaces of the sigma models; in Section III, the contents of the diagram (1) are developed, presenting the geometric description of the PLT-duality. The dynamical questions are addressed in Sec-

tion IV, dealing with the hamiltonian and lagrangian descriptions of the PLT-dual models. In Section V, we illustrate the construction for the Abelian and non-Abelian duality, giving the explicit duality transformations and identifying the dualizable subspaces. Finally, some conclusion and comments are condensed in the last Section.

I- THE WZW MODEL PHASE SPACE GEOMETRY

The WZW model was proposed by Witten [15] as a modification of the principal sigma model driving to equation of motion admitting factorizable general solutions: $g(\sigma, t) = g_l(\sigma + t)g_r(\sigma - t)$ or $g(\sigma, t) = g_r(\sigma - t)g_l(\sigma + t)$. This is attained by adding to the original action of the sigma model the Wess-Zumino term, and the order of the light cone factors in $g(\sigma, t)$ depends on the sign of the added term.

As it is well known, the phase space of a sigma model with target space the group manifold G is the cotangent bundle T^*LG of the loop group LG that turns to be a symplectic manifold with the canonical symplectic form ω_o [17], and the dynamics is determined by the election of the Hamiltonian function. However, there is no election of Hamiltonian function on (T^*LG, ω_o) driving to equations of motion equivalent to the WZW ones. In fact, as shown in ref. [18], the addition of Wess-Zumino term amounts to a modification of the canonical Poisson brackets on T^*LG . Its symplectic counterpart is exhaustively studied in references [16],[26] and references therein, where a cocycle extension of the canonical symplectic form ω_o is considered in combination with the Marsden-Weinstein reduction procedure in order to recover the WZW equation of motion. This last description provides the framework for our approach to Poisson-Lie T-duality, so it is worthwhile to briefly review it below.

Let us consider a connected Lie group D and its loop group LD . For $l \in LD$, l' denotes the derivative in the loop parameter $\sigma \in S^1$, and we write vl^{-1} and $l^{-1}v$ for the right and left translation of any vector field $v \in TD$. Let \mathfrak{d} be the Lie algebra of D equipped with a non degenerate symmetric Ad -invariant bilinear form $(\cdot, \cdot)_{\mathfrak{d}}$. Frequently we will work with the subset $L\mathfrak{d}^* \subset (L\mathfrak{d})^*$ instead of $(L\mathfrak{d})^*$ itself, and we identify it with $L\mathfrak{d}$ through the map $\psi : L\mathfrak{d} \rightarrow L\mathfrak{d}^*$ provided by the bilinear form

$$(\cdot, \cdot)_{L\mathfrak{d}} \equiv \frac{1}{2\pi} \int_{S^1} (\cdot, \cdot)_{\mathfrak{d}}$$

on $L\mathfrak{d}$. This bilinear form defines a 2-cocycle $\Gamma_k : L\mathfrak{d} \times L\mathfrak{d} \rightarrow \mathbb{R}$ [14],

$$\Gamma_k(X, Y) = \frac{k}{2\pi} \int_{S^1} (X(\sigma), Y'(\sigma))_{\mathfrak{d}} d\sigma$$

with $X, Y \in L\mathfrak{d}$. It is invariant under the action of the orientation preserving diffeomorphism group of the circle, $Diff^+(S^1)$, and invariant under the adjoint action of constant loops, $\Gamma_k(Ad_{l_o}X, Ad_{l_o}Y) = \Gamma_k(X, Y)$, for $l_o \in D$. It can be derived from the 1-cocycle $C_k : LD \rightarrow L\mathfrak{d}^*$,

$$C_k(l) = k\psi(l'l^{-1}) \quad .$$

We identify the cotangent bundle T^*LD with $LD \times (L\mathfrak{d})^*$ by left translation and, in practice, we shall work on $L(D \times \mathfrak{d}^*)$. The pair (T^*LD, ω_o) , where ω_o is the canonical 2-form defined as [17]

$$\langle \omega_o, (v, \rho) \otimes (w, \xi) \rangle_{(l, \varphi)} = -\langle \rho, l^{-1}w \rangle_{L\mathfrak{d}} + \langle \xi, l^{-1}v \rangle_{L\mathfrak{d}} + \langle \varphi, [l^{-1}v, l^{-1}w] \rangle_{L\mathfrak{d}}$$

for $(v, \rho), (w, \lambda) \in T_{(l, \varphi)}L(D \times \mathfrak{d}^*)$, is the symplectic manifold on which sigma models with targets D are framed on. As explained above, the WZW model doesn't fit this symplectic structure. Indeed, the symplectic manifold underlying the chiral sectors of the WZW model is (T^*LD, ω_Γ) , with ω_Γ being a symplectic 2-form obtained by adding a cocycle term to ω_o ,

$$\langle \omega_\Gamma, \lambda_*^{-1}(v, \rho) \otimes \lambda_*^{-1}(w, \xi) \rangle_{(l, \varphi)} = \langle \omega_o, (v, \rho) \otimes (w, \xi) \rangle_{(l, \varphi)} - \Gamma_k(vl^{-1}, wl^{-1})$$

for $(v, \rho), (w, \lambda) \in T_{(l, \varphi)}L(D \times \mathfrak{d}^*)$. This symplectic structure has also a natural interpretation in terms of symplectic groupoids [35] for the underlying infinite dimensional affine Poisson algebra (for details see [33]). Indeed, the cocycle Γ_k defines an Affine Poisson structure on $L\mathfrak{d}^*$ induced by the action groupoid $H = LD \triangleright L\mathfrak{d}^* \rightrightarrows L\mathfrak{d}_{Aff}^*$, with LD acting by the (right) affine coadjoint action $A_l(\xi) = Ad_l^*\xi + C_k(l^{-1})$, and supplied with the symplectic form

$$\omega_\Gamma^R = \omega_o^R - \Gamma_k(dll^{-1} \otimes dll^{-1})$$

where ω_o^R is the symplectic form on $L(D \times \mathfrak{d}^*)$ obtained from the standard one on T^*LD trivialized by right translations. So we see that ω_Γ^R becomes the above introduced ω_Γ under the diffeomorphism $(l, \varphi) \longrightarrow (l, Ad_{l^{-1}}^*\varphi)$ which switches from left to right trivialization of T^*LD .

Observe that ω_Γ it is no longer a bi-invariant 2-form, only the invariance under right translation $\varrho_m^{L(D \times \mathfrak{d}^*)}(l, \mu) = (lm^{-1}, Ad_{m^{-1}}^* \mu)$, $m \in LD$, remains. Tied to it there is a non Ad -equivariant momentum map $J^R : L(D \times \mathfrak{d}^*) \rightarrow L\mathfrak{d}^*$,

$$J^R(l, \xi) = -\xi + k\psi(l^{-1}l')$$

with associated 1-cocycle $-C_k$, so that $J^R\left(\varrho_m^{L(D \times \mathfrak{d}^*)}(l, \xi)\right) - Ad_{m^{-1}}^* J^R(l, \xi) = -k\psi(k'k^{-1})$ and J^R is a Poisson map to $L\mathfrak{d}_{Aff}^*$.

When the corresponding central extension LD^\wedge of LD does exist, ω_Γ can be obtained from the standard symplectic structure on $T^*LD^\wedge \stackrel{Left}{\cong} LD^\wedge \times (L\mathfrak{d}_\Gamma)^*$ by reduction under the corresponding $S^1 \subset LD^\wedge$ action and the Ad -equivariance of J^R is then restored substituting $L\mathfrak{d}$ by the centrally extended Lie algebra $L\mathfrak{d}_\Gamma$, defined by the cocycle Γ_k . The centrally extended adjoint and coadjoint actions of LD^\wedge on $L\mathfrak{d}_\Gamma$ and $L\mathfrak{d}_\Gamma^*$ are defined as

$$\begin{aligned}\widehat{Ad}_l(X, a) &= (Ad_l X, a + k \langle \psi(l'l^{-1}), X \rangle) \\ \widehat{Ad}_{l^{-1}}^*(\xi, b) &= (Ad_{l^{-1}}^* \xi + bk\psi(l'l^{-1}), b)\end{aligned}$$

Note that the $S^1 \subset LD^\wedge$ action is trivial and the embedding $\xi \mapsto (\xi, 1)$ is a Poisson map from $L\mathfrak{d}_{Aff}^*$ to $L\mathfrak{d}_\Gamma \sim L\mathfrak{d} \times \mathbb{R}$ which maps the *affine* coadjoint action of LD to the centrally extended one of $LD \hookrightarrow LD^\wedge$. Now, the extended momentum map $\hat{J}^R : L(D \times \mathfrak{d}^*) \rightarrow L\mathfrak{d}_\Gamma^*$ is

$$\hat{J}^R(l, \xi) = (J^R(l, \xi), 1) = (k\psi(l^{-1}l') - \xi, 1) \quad (2)$$

which is \widehat{Ad}^{LD} -equivariant, $\hat{J}^R\left(\varrho_m^{L(D \times \mathfrak{d}^*)}(l, \xi)\right) - \widehat{Ad}_{m^{-1}}^{LD*} \hat{J}^R(l, \xi) = (0, 0)$.

The next step is to apply the Marsden-Weinstein reduction procedure [20] to the point $(0, 1) \in L\mathfrak{d}_\Gamma^*$. The restriction of ω_Γ to $\left(\hat{J}^R\right)^{-1}(0, 1)$ defines the degenerate 2-form

$$\gamma(v, w) = \Gamma(l^{-1}v, l^{-1}w) \quad (3)$$

with null distribution generated by the infinitesimal action of constant loops D . In fact, in order to obtain the reduced space, $\left(\hat{J}^R\right)^{-1}(0, 1)$ must be quotiented by the stabilizer of $(0, 1) \in L\mathfrak{d}_\Gamma^*$, that is, by the subgroup of constant loops, $Stab(0, 1) = D$. Since $\left(\hat{J}^R\right)^{-1}(0, 1) = \{(l, k\psi(l^{-1}l')) \mid l \in LD\} \cong LD$, the reduced space can be identified with the subgroup of *based loops*

$$\frac{\left(\hat{J}^R\right)^{-1}(0)}{D} \equiv \Omega D = \{[l] = ll^{-1}(0) \mid l \in LD\}.$$

so that the fibration

$$\Lambda : LD \rightarrow \Omega D \quad / \quad \Lambda([l]) = [l] \quad , \quad (4)$$

with fiber D , provides the symplectic 2-form $\omega_{\Omega D}$ on ΩD defined from $\Lambda^* \omega_{\Omega D} = \gamma$.

After the reduction procedure, $\omega_{\Omega D}$ is still invariant under the residual left action of LD on ΩD

$$LD \times \Omega D \rightarrow \Omega D \quad / \quad (m, [l]) \longrightarrow [ml] \quad (5)$$

The associated momentum map $\Phi : \Omega D \rightarrow L\mathfrak{d}^* / \Phi([l]) = k\psi(l'l^{-1})$ is not Ad -equivariant. Again, introducing the extended momentum map

$$\hat{\Phi} : \Omega D \rightarrow L\mathfrak{d}_\Gamma^* \quad / \quad \hat{\Phi}([l]) = (\Phi([l]), 1) = \widehat{Ad}_{l^{-1}}^{LD*}(0, 1) \quad , \quad (6)$$

the equivariance is restored and the vertical arrow in diagram (1) is explained. In fact, remember that $L\mathfrak{d}_\Gamma^*$ is a Poisson manifold with Poisson bracket $\{f, g\}_{KK}(\eta) = \langle \eta, [df, dg]_{L\mathfrak{d}_\Gamma} \rangle$ and their symplectic leaves are the coadjoint orbits. Thereby, $\hat{\Phi}$ becomes into a symplectic map (local diffeomorphism) onto the pure central extension orbit $\mathcal{O}_{(0,1)} \equiv \mathcal{O}$, $\hat{\Phi} : (\Omega D, \omega_{\Omega D}) \rightarrow (\mathcal{O}, \omega_{KK})$, with ω_{KK} being the Kirillov-Kostant symplectic form that on the \mathcal{O} reduces to

$$\left\langle \omega_{KK}, \widehat{ad}_X^*(k\psi(l'l^{-1}), 1) \otimes \widehat{ad}_Y^*(k\psi(l'l^{-1}), 1) \right\rangle_{(k\psi(l'l^{-1}), 1)} = \Gamma_k(X, Y)$$

for $X, Y \in (L\mathfrak{d}/\mathfrak{d})_\Gamma$. Then, for any vector $[v] \in T_{[l]}\Omega D$, one has $\left(\hat{\Phi}\right)_* [v] = -ad_{vl^{-1}}^* Ad_{l^{-1}}^*(0, 1)$ and

$$\left\langle \left(\hat{\Phi}\right)^* \omega_{KK}, [v] \otimes [w] \right\rangle_{[l]} = \Gamma(l^{-1}v, l^{-1}w) = \langle \omega_{\Omega D}, [v] \otimes [w] \rangle$$

It is worth remarking that only the coadjoint orbit through the pure central extension element $(0, a)$, namely $(\mathcal{O}_{(0,a)}, \omega_{KK})$, is (locally) symplectomorphic to $(\Omega D, \omega_{\Omega D})$.

II- HAMILTONIAN LD ACTIONS ON DUAL PHASE-SPACES

In the following subsections we shall introduce LD actions on the sigma model phase spaces LTG and LTG^* for G and G^* being dual Poisson-Lie groups composing a (connected, simply connected) perfect Drinfeld double D , i.e., it admits a global factorization $D = G \bowtie G^*$. Under this conditions we have the exact sequences

$$\begin{aligned} 0 &\longrightarrow \mathfrak{g} \longrightarrow \mathfrak{d} \longrightarrow \mathfrak{g}^* \longrightarrow 0 \\ 0 &\longrightarrow G \longrightarrow D \longrightarrow G^* \longrightarrow 0 \end{aligned}$$

where $\mathfrak{d} = \mathfrak{g} \bowtie \mathfrak{g}^*$ is the Lie bialgebra of D supplied with the symmetric invariant non degenerate bilinear form $(\cdot, \cdot)_{\mathfrak{d}}$ given by the pairing between \mathfrak{g} and \mathfrak{g}^* , so they are *isotropic subspaces* in relation to it. Identifying \mathfrak{g}^* with the Lie algebra of G^* , we can have the embedding

$$L(G \times \mathfrak{g}^*) \xrightarrow{\iota_G} LD \times L\mathfrak{d}^* \quad / \quad (g, \alpha) \longmapsto (g, Ad_{g^{-1}}^{LD*} \psi(\alpha) + C_k(g))$$

and define the map $\mu : L(G \times \mathfrak{g}^*) \longrightarrow L\mathfrak{d}_{Aff}^*$ given by the diagram

$$\begin{array}{ccc} L(G \times \mathfrak{g}^*) & \xrightarrow{i_G} & LD \times L\mathfrak{d}^* \\ & \searrow \mu & \downarrow s \\ & & L\mathfrak{d}_{Aff}^* \end{array}$$

where $s(l, \xi) = \xi$ is a Poisson map (in fact, it is the source map for the symplectic groupoid $H = LD \triangleright L\mathfrak{d}^* \rightrightarrows L\mathfrak{d}_{Aff}^*$). From the isotropy of \mathfrak{g} with respect to $(\cdot, \cdot)_{\mathfrak{d}}$, it can be easily seen that $\iota_G^* \omega_{\Gamma}^R = \omega_o^{LG}$ where ω_o^{LG} is the standard symplectic structure on $LT^*G \sim L(G \times \mathfrak{g}^*)$ trivialized by left translations. So μ is a *Poisson map* for this symplectic structure on $L(G \times \mathfrak{g}^*)$, and an analogous construction can be repeated on the dual group giving the *Poisson map* $\tilde{\mu} : L(G^* \times \mathfrak{g}) \longrightarrow L\mathfrak{d}_{Aff}^*$. These maps can be regarded as giving symplectic realizations of $L\mathfrak{d}_{Aff}^*$ and, as we shall see in the next sections, they give (non equivariant) momentum maps for LD actions on $L(G \times \mathfrak{g}^*)$ and $L(G^* \times \mathfrak{g})$. For simplicity, the centrally extended loop group LD^\wedge is assumed to exist, so $(\mu, 1)$ and $(\tilde{\mu}, 1)$ give the usual equivariant momentum maps for the corresponding LD^\wedge actions, however we remark that all the following constructions can be performed also without using LD^\wedge at all, just replacing $L\mathfrak{d}_{\Gamma}^*$ by $L\mathfrak{d}_{Aff}^*$.

Hamiltonian LD^\wedge action on the G -sigma model phase space

In this section we introduce a LD^\wedge symmetry on the sigma model with target G . One of the most striking features of the double Lie groups and Lie bialgebras is the existence of reciprocal actions between the factors G and G^* called *dressing actions* [3, 19],[3]. Writing every element $l \in D$ as $l = g\tilde{h}$, with $g \in G$ and $\tilde{h} \in G^*$, the product $\tilde{h}g$ in D can be written as $\tilde{h}g = g^{\tilde{h}}\tilde{h}^g$, with $g^{\tilde{h}} \in G$ and $\tilde{h}^g \in G^*$. The dressing action of G^* on G is then defined as

$$\text{Dr} : G^* \times G \longrightarrow G \quad / \quad \text{Dr}(\tilde{h}, g) = g^{\tilde{h}} \quad (7)$$

The infinitesimal generator of this action in the point $g \in G$ is, for $\xi \in \mathfrak{g}^*$,

$$\xi \longrightarrow \mathbf{dr}(\xi)_g = - \left. \frac{d}{dt} \mathbf{Dr}(e^{t\xi}, g) \right|_{t=0}$$

such that, for $\eta \in \mathfrak{g}^*$, we have $[\mathbf{dr}(\xi)_g, \mathbf{dr}(\eta)_g] = \mathbf{dr}([\xi, \eta]_{\mathfrak{g}^*})_g$. It satisfy the relation

$$Ad_{g^{-1}}^D \xi = - (L_{g^{-1}})_* \mathbf{dr}(\xi)_g + Ad_g^* \xi \quad (8)$$

where $Ad_{g^{-1}}^D \xi \in \mathfrak{g} \ltimes \mathfrak{g}^*$ is the adjoint action of D . Then, we can write $\mathbf{dr}(\xi)_g = - (L_g)_* \Pi_{\mathfrak{g}} Ad_{g^{-1}}^D \xi$, with $\Pi_{\mathfrak{g}} : \mathfrak{g} \oplus \mathfrak{g}^* \rightarrow \mathfrak{g}$ being the projector.

From this dressing action we build up a symplectic action of the double LD^\wedge on T^*LG whose momentum map furnish the arrow μ in diagram (1). First, we introduce the map $\mathbf{d}^{LG} : LD \times LG \longrightarrow LG$ defined as

$$\mathbf{d}^{LG}(a\tilde{b}, g) = ag^{\tilde{b}} \quad (9)$$

for $a, g \in LG$ and $\tilde{b} \in LG^*$, which is a left action. It can be lifted to the left trivialization of LT^*G , namely $L(G \times \mathfrak{g}^*)$, and then promoted to an action of the centrally extended double $LD^\wedge \simeq LD \times \mathbb{T}^1$, as explained in the following proposition.

Proposition: *The map $\hat{\mathbf{d}} : LD^\wedge \times L(G \times \mathfrak{g}^*) \longrightarrow L(G \times \mathfrak{g}^*)$,*

$$\hat{\mathbf{d}}(a\tilde{b}, (g, \eta)) = \left(ag^{\tilde{b}}, Ad_{\tilde{b}^g}^{LD} \eta + k(\tilde{b}^g)'(\tilde{b}^g)^{-1} \right) \quad (10)$$

is a left symplectic action, with \widehat{Ad}^{LD} -equivariant momentum mapping

$$\mu(g, \eta) = \widehat{Ad}_{g^{-1}}^{LD*}(\psi(\eta), 1) \quad (11)$$

Proof: In order to obtain $\hat{\mathbf{d}}$, we lift the action \mathbf{d}^{LG} , eq. (9), to T^*LG . In doing so, we consider the associated map $\mathbf{d}_{ab}^{LG} : LG \longrightarrow LG$ such that $\mathbf{d}_{ab}^{LG}(g) \equiv \mathbf{d}^{LG}(a\tilde{b}, g)$. Its differential is $(\mathbf{d}_{ab}^{LG})_{*g} v = (L_{ag^{\tilde{b}}})_* Ad_{(\tilde{b}^g)^{-1}}^{LG*}(g^{-1}v_g)$ for any tangent vector $v \in T_g G$, from where we obtain the pullback on a 1-forms α in the transformed point $ag^{\tilde{b}}$

$$(\mathbf{d}_{ab}^{LG})^*(ag^{\tilde{b}}, \alpha) = \left(g, (L_{g^{-1}})^* Ad_{(\tilde{b}^g)^{-1}}^{LG*} (L_{ag^{\tilde{b}}})^* \alpha \right)$$

In body coordinates $LG \times L\mathfrak{g}^*$, and after a change of variables, we get $\mathbf{d} : LD \times L(G \times \mathfrak{g}^*) \longrightarrow L(G \times \mathfrak{g}^*)$

$$\mathbf{d}(a\tilde{b}, (g, \eta)) = (ag^{\tilde{b}}, Ad_{\tilde{b}^g}^{LD} \eta) \quad (12)$$

which is a well defined action. The momentum map $\mu_o : L(G \times \mathfrak{g}^*) \longrightarrow L\mathfrak{d}^*$ associated to this action is easily calculated from the infinitesimal generator of the action (9)

$$(\mathbf{d}_g^{LG})_{*e}(X, \xi) = (R_g)_* X - \mathbf{dr}(\xi)_g$$

for any $(X, \xi) \in L\mathfrak{d} = L(\mathfrak{g} \bowtie \mathfrak{g}^*)$, with $\mathbf{d}_g^{LG}(a\tilde{b}) \equiv \mathbf{d}^{LG}(a\tilde{b}, g)$. Using the expression (8), and by the identification $\psi : L\mathfrak{d} \rightarrow L\mathfrak{d}^*$ provided by the bilinear form $(\cdot, \cdot)_{L\mathfrak{d}}$, we have

$$\mu_o(g, \eta) = \psi(Ad_g^{LD}\eta)$$

which obviously is Ad^{LD} -equivariant since it is associated to the lift to the cotangent bundle of an action on LG .

The action \mathbf{d} is promoted to an action of the centrally extended double $LD^\wedge \simeq LD \times \mathbb{T}^1$, $\hat{\mathbf{d}}^{L(G \times \mathfrak{g}^*)} : LD^\wedge \times L(G \times \mathfrak{g}^*) \longrightarrow L(G \times \mathfrak{g}^*)$ by the definition

$$\hat{\mathbf{d}}\left((a\tilde{b}, \theta), (g, \eta)\right) = \left(ag^{\tilde{b}}, Ad_{\tilde{b}g}^{LD}\eta + k(\tilde{b}^g)'(\tilde{b}^g)^{-1}\right)$$

where the element $\theta \in \mathbb{T}^1$ acts trivially, so it descends to an LD action and by this reason it is omitted in (10). It is worth remarking that $\hat{\mathbf{d}}$ is no longer a lift of a transformation on LG .

The infinitesimal action of some $(X, \xi) \in L(\mathfrak{g} \oplus \mathfrak{g}^*)$ at the point $(g, \eta) \in L(G \times \mathfrak{g}^*)$ is straightforwardly computed, giving

$$\left(\hat{\mathbf{d}}_{(g, \eta)}\right)_{*e}(X, \xi) = \left((R_g)_* X - \mathbf{dr}(\xi)_g, [Ad_g^{LG*}\xi, \eta]_{L\mathfrak{g}^*} + Ad_g^{LG*}ad_{g'g^{-1}}^{LD*}\xi + Ad_g^{LG*}\xi'\right) \quad , \quad (13)$$

and from this expression we calculate the momentum map $\mu : LG \times L\mathfrak{g}^* \longrightarrow L(\mathfrak{g} \bowtie \mathfrak{g}^*)_\Gamma^*$ using the canonical symplectic structure ω_o on $L(G \times \mathfrak{g}^*)$, obtaining

$$\mu(g, \eta) = (Ad_{g^{-1}}^{LD*}\psi(\eta) + k\psi(g'g^{-1}), 1) = \widehat{Ad}_{g^{-1}}^{LD*}(\psi(\eta), 1)$$

that satisfy the \widehat{Ad} -equivariance relation

$$\mu\left(\hat{\mathbf{d}}\left(a\tilde{b}, (g, \eta)\right)\right) = \widehat{Ad}_{(a\tilde{b})}^{LD*}\mu(g, \eta)$$

Obviously, since $\left(\hat{\mathbf{d}}_{(g, \eta)}\right)_*(X, \xi)$ are Hamiltonian for all $(X, \xi) \in L\mathfrak{d}_\Gamma$, the Lie derivative $\mathbf{L}_{\left(\hat{\mathbf{d}}_{(g, \eta)}\right)_*(X, \xi)}\omega_o = 0$ meaning that $\hat{\mathbf{d}}^{L(G \times \mathfrak{g}^*)}$ leaves the canonical symplectic form invariant. ■

Now, some remarks are in order. First, note that $\hat{\mathbf{d}}$ is not a *free* action, the subgroup G^* leaves invariant the point $(e, 0)$. Then, observe that if the central extension of the loop

group does not exist the above proposition still defines a hamiltonian LD action. We can proceed in an analogous manner by using the affine Poisson structure and affine coadjoint LD actions. Finally, $(T^*LG, \omega_o, \hat{\mathfrak{d}}, \mu)$ is a *Hamiltonian LD^\wedge -space*, with μ equivariant and the image through μ of T^*LG is a union of coadjoint orbits in $L\mathfrak{d}_\Gamma^*$.

Factorizing $\hat{\Phi} : \Omega D \rightarrow \mathcal{O}(0,1)$ through LT^*G

In this section we shall show that $\hat{\Phi} : \Omega D \rightarrow \mathcal{O} \subset L\mathfrak{d}_\Gamma^*$ can be factorized through $\mu : LT^*G \rightarrow L\mathfrak{d}_\Gamma^*$ on the pure central extension coadjoint orbit, composing a three vertices commutative diagram like the left triangle of (1).

By definition $\mathcal{O} \equiv \mathcal{O}_{(0,1)} = \left\{ \widehat{Ad}_{(ab)}^{LD*} (0,1) \mid ab \in LD \right\}$ and any point $(g, \eta) \in \mu^{-1}(\mathcal{O})$ is, due to the equivariance of μ , of the form $(g, \eta) = \hat{\mathfrak{d}}(a\tilde{b}, (e, 0))$ for some $a\tilde{b} \in LD$ implying that $\mu^{-1}(\mathcal{O})$ is just the orbit of LD through the point $(e, 0) \in L(G \times \mathfrak{g}^*)$. In terms of the orbit map $\hat{\mathfrak{d}}_{(e,0)} : LD \longrightarrow L(G \times \mathfrak{g}^*)$,

$$\hat{\mathfrak{d}}_{(e,0)}(a\tilde{b}) = \hat{\mathfrak{d}}(a\tilde{b}, (e, 0)) = (a, k\tilde{b}'\tilde{b}^{-1})$$

we write

$$\mu^{-1}(\mathcal{O}) = \text{Im } \hat{\mathfrak{d}}_{(e,0)}$$

Hence, the tangent space of this LD -orbit is spanned by the infinitesimal generators of the action $\hat{\mathfrak{d}}$, given in eq. (13), for every $(g, \eta) = (a, k\tilde{b}'\tilde{b}^{-1}) \in \mu^{-1}(\mathcal{O})$, and it can be split in the direct sum $\hat{\mathfrak{d}}(\Lambda_* L\mathfrak{d}) \oplus \hat{\mathfrak{d}}(\widehat{Ad}_{ab}^{LD}\mathfrak{d})$. In fact, for any tangent vector (V, ξ) to that point there exist some $[X] \in \Lambda_* L\mathfrak{d}$ and $X_o \in \mathfrak{d}$ such that

$$(V, \xi)_{(g, \eta)} = \left(\hat{\mathfrak{d}}_{(g, \eta)} \right)_{*e}([X]) + \left(\hat{\mathfrak{d}}_{(g, \eta)} \right)_{*e}(\widehat{Ad}_{ab}^{LD} X_o) \quad (14)$$

Observe that $\left(\hat{\mathfrak{d}}_{(e,0)} \right)^* \omega_o = \gamma$, eq. (3) and, beside to the fact that $\omega_o|_{\mu^{-1}(\mathcal{O})} = \mu^* \omega_{KK}$, it amounts to a presymplectic submersion $\mu \circ \hat{\mathfrak{d}}_{(e,0)} : (LD, \gamma) \longrightarrow (\mathcal{O}, \omega_{KK})$.

Theorem: *Let $\omega_o|_{\mu^{-1}(\mathcal{O})}$ the restriction of ω_o to $\mu^{-1}(\mathcal{O})$. Then, its null distribution is spanned by the infinitesimal generator of subgroup $\widehat{Ad}_{ab}^{LD} D$ with leaf through $(a, k\tilde{b}'\tilde{b}^{-1}) \in \mu^{-1}(\mathcal{O})$ being $\mu^{-1}(\widehat{Ad}_{(ab)}^{LD*}(0,1)) = \hat{\mathfrak{d}}^{L(G \times \mathfrak{g}^*)}([a\tilde{b}], (G, 0))$, so $\mu^{-1}(\mathcal{O}) \longrightarrow \mathcal{O}$ is a fibration with $\dim \mathfrak{g}$ -dimensional fibers. Moreover, their symplectic leaves are the orbits of ΩD by the action $\hat{\mathfrak{d}}$.*

Proof: The isotropy group of a point $\widehat{Ad}_{(a\tilde{b})}^{LD*}(0, 1) \in \mathcal{O}$ is $Ad_{a\tilde{b}}^{LD}D$, and its infinitesimal action pulled-back by μ gives rise to the null distribution of $\omega_o|_{\mu^{-1}(\mathcal{O})}$. So, we have the null foliation with leaf through $\hat{\mathfrak{d}}(a\tilde{b}, (e, 0))$ being the orbits of the subgroup $Ad_{a\tilde{b}}^{LD}D$ and of dimension $\dim \mathfrak{g}$, as it can be seen from the relation

$$\hat{\mathfrak{d}}^{L(G \times \mathfrak{g}^*)}((a\tilde{b})l_o(a\tilde{b})^{-1}, (a, k\tilde{b}'\tilde{b}^{-1})) = \hat{\mathfrak{d}}^{L(G \times \mathfrak{g}^*)}((a\tilde{b})l_o, (e, 0)) = \hat{\mathfrak{d}}^{L(G \times \mathfrak{g}^*)}((a\tilde{b}), (g_o, 0))$$

for all $l_o = g_o\tilde{h}_o \in D$ or, infinitesimally, since for $(g, \eta) = \hat{\mathfrak{d}}(a\tilde{b}, (e, 0))$,

$$\left(\hat{\mathfrak{d}}_{(g, \eta)}\right)_{*e}(Ad_{a\tilde{b}}^{LD}X_o) = \left(\hat{\mathfrak{d}}_{\hat{\mathfrak{d}}(a\tilde{b}, (e, 0))}\right)_{*e}(Ad_{a\tilde{b}}^{LD}X_o) = \left(\hat{\mathfrak{d}}_{a\tilde{b}}\right)_{*(e, 0)}\left(\hat{\mathfrak{d}}_{(e, 0)}\right)_{*e}X_o$$

and from (13) $\left(\hat{\mathfrak{d}}_{(e, 0)}\right)_{*e}X_o = (\Pi_{\mathfrak{g}}(X_o), 0)$ for all $X_o \in \mathfrak{d}$.

The complementary distribution constitutes then the symplectic foliation of $\omega_o|_{\mu^{-1}(\mathcal{O})}$, and it is spanned by the second term in the direct sum (14) with leaves being the orbits of ΩD in $\mu^{-1}(\mathcal{O})$.

All this can be obtained from a more general result contained in a theorem by Kazhdan, Kostant and Sternberg [22] (see Theorem 26.2 in ref. [21]). ■

A further consequence is that any point $(g, \eta) \in \mu^{-1}(\mathcal{O})$ can be characterized by a pair $\left([a\tilde{b}], g_o\right) \in \Omega D \times G$ such that $(g, \eta) = \hat{\mathfrak{d}}^{L(G \times \mathfrak{g}^*)}\left([a\tilde{b}], (g_o, 0)\right)$. Thinking of the composition

$$\mu^{-1}(\mathcal{O}) \xrightarrow{\mu} \mathcal{O} \xrightarrow{\hat{\Phi}^{-1}} \Omega D$$

as a fibration, the direct sum (14) defines a connection with the horizontal subspace being the orbits $S(g_o)$ of ΩD through the point $(g_o, 0)$, for each $g_o \in G$. Then, we may consider a section $\zeta : \Omega D \rightarrow \mu^{-1}(\mathcal{O})$ with equivariance property

$$\zeta(l \cdot [m]) = \hat{\mathfrak{d}}(l, \zeta([m])) \quad ,$$

to obtain a factorization of $\hat{\Phi}$ through T^*LG ,

$$\begin{array}{ccc} & & (L\mathfrak{d}^*; \{, \}_{KK}) \\ & \nearrow \mu & \uparrow \hat{\Phi} \\ (T^*LG; \omega_o) & & (\Omega D; \omega_{\Omega D}) \\ & \searrow \varnothing & \end{array} \quad (15)$$

with arrows being presymplectic maps, reproducing the left triangle of diagram (1).

In order to get a symplectic factorization, we define a family $\{\varsigma_{g_o}\}_{g_o \in G}$ of *horizontal* sections with each image being a symplectic leaf $S(g_o) \subset \mu^{-1}(\mathcal{O})$, the label g_o being the point $(g_o, 0)$ in $\mu^{-1}(\mathcal{O})$ they pass through

$$\begin{aligned}\varsigma_{g_o} : \Omega D &\longrightarrow S(g_o) \subset \mu^{-1}(\mathcal{O}) \\ \varsigma_{g_o}([l]) &= \hat{\mathbf{d}}([l], (g_o, 0))\end{aligned}$$

for $g_o \in G$. Indeed, they are horizontal sections if one regards (14) as defining a connection on the trivial G -bundle structure in $\mu^{-1}(\mathcal{O})$.

Proposition: *Let $\varsigma_{g_o} : (\Omega D, \omega_{\Omega D}) \longrightarrow (S(g_o), \omega_o|_{S(g_o)})$ is a symplectic map, for any $g_o \in G$.*

Proof: We have to show that $\omega_{\Omega D} = \varsigma_{g_o}^* \omega_o|_{S(g_o)}$. Let $v \in T_l LD$, and $[v] = \Lambda_{*l} v \in T_{[l]} \Omega D$ a tangent vector to the point $[l] \in \Omega D$. Then

$$(\varsigma_{g_o})_{*[l]}([v]_b) = \left(\hat{\mathbf{d}}_{\varsigma_{g_o}([l])} \right)_* ([v] [l]^{-1})$$

where $\hat{\mathbf{d}}_{\varsigma_{g_o}([l])} : LD \longrightarrow T^*LG$ is the induced map by the action $\hat{\mathbf{d}}$ describing the orbit of LD through $(g_o, 0) \in L(G \times \mathfrak{g}^*)$. Using the equivariant momentum map μ and having in mind that the stabilizer of the point $\widehat{Ad}_{[ab]^{-1}}^{LD*}(0, 1)$ is $Ad_{ab}^{LD*}D$, we conclude that

$$\begin{aligned}\left\langle \omega_o|_{\mu^{-1}(\mathcal{O})}, (\varsigma_{g_o})_{*[l]}([v]) \otimes (\varsigma_{g_o})_{*[l]}([w]) \right\rangle &= \left\langle Ad_{l^{-1}}^{LD*}(0, 1), [v] [l]^{-1}, [w] [l]^{-1} \right\rangle_{L\mathfrak{d}\Gamma} \\ &= \Gamma_k([l]^{-1} [v], [l]^{-1} [w])\end{aligned}$$

showing that $(\varsigma_{g_o})^* \omega_o|_{S_R(g_o)} = \omega_{\Omega D}$. ■

Then, the diagram (15) can be refined to the following commutative diagram with arrows being symplectic maps:

$$\begin{array}{ccc} & & (\mathcal{O}; \omega_{KK}) \\ & \nearrow \mu & \uparrow \hat{\Phi} \\ (\mu^{-1}(\mathcal{O}); \omega_o) & & (\Omega D; \omega_{\Omega D}) \\ & \searrow \varsigma_{g_o} & \end{array} \quad (16)$$

for each $g_o \in G$.

Hamiltonian LD^\wedge action on the G^* -sigma model phase space

Because of the symmetric role played by G and G^* in the double D , all the results obtained above can be straightforwardly dualized just interchanging their roles. In spite of this general principle, a few details and notation are in order.

Let us consider D with the opposite factorization, denoted as $D \rightarrow D^T = G^* \bowtie G$, so that every element is now written as $\tilde{h}g$ with $\tilde{h} \in G^*$ and $g \in G$. Then, for $g \in G$ and $\tilde{h} \in G^*$ there exist $\tilde{h}_g \in G^*$ and $g_{\tilde{h}} \in G$ such that $g\tilde{h} = \tilde{h}_g g_{\tilde{h}}$. This factorization relates with the opposite one by $\tilde{b}_a = \left((\tilde{b}^{-1})^{a^{-1}} \right)^{-1}$ and $a_{\tilde{b}} = \left((a^{-1})^{\tilde{b}^{-1}} \right)^{-1}$. The dressing action $\widetilde{\text{Dr}} : G \times G^* \rightarrow G^*$ is $\widetilde{\text{Dr}}(g, \tilde{h}) = \tilde{h}_g$ and, by composing it with the right action of LG^* on itself, we get the action $\mathbf{b}^{LG^*} : LD \times LG^* \rightarrow LG^*$ defined as $\mathbf{b}^{LG^*}(\tilde{b}a, \tilde{h}) = \tilde{b}\tilde{h}_a$ with $a \in LG$ and $\tilde{h}, \tilde{b} \in LG^*$.

For a sigma model on the target G^* , the phase space is the symplectic manifold $(T^*LG^*, \tilde{\omega}_o)$. Then, we lift \mathbf{b}^{LG^*} to the left trivialization of $T^*LG^* \cong L(G^* \times \mathfrak{g})$, and promote it to an extended symmetry $\hat{\mathbf{b}} : LD^\wedge \times L(G^* \times \mathfrak{g}) \rightarrow L(G \times \mathfrak{g}^*)$,

$$\hat{\mathbf{b}}\left(\tilde{b}a, \left(\tilde{h}, Z\right)\right) = \left(\tilde{b}\tilde{h}_a, \text{Ad}_{a_{\tilde{h}}}^{LD} Z + k(a_{\tilde{h}})'(a_{\tilde{h}})^{-1}\right) \quad (17)$$

with an $\widehat{\text{Ad}}^{LD}$ -equivariant momentum map

$$\tilde{\mu}\left(\tilde{h}, Z\right) = \widehat{\text{Ad}}_{\tilde{h}^{-1}}^{LD*}(\psi(Z), 1) = \left(\psi\left(\text{Ad}_{\tilde{h}}^{LD} Z + k\tilde{h}'\tilde{h}^{-1}\right), 1\right) \quad (18)$$

In terms of the orbit map $\hat{\mathbf{b}}_{(e,0)} : LD \rightarrow L(G^* \times \mathfrak{g})$ associated to the action $\hat{\mathbf{b}}$, eq. (17),

$$\hat{\mathbf{b}}_{(e,0)}(a\tilde{b}) = \hat{\mathbf{b}}(\tilde{b}a, (e, 0)) = (\tilde{b}, k a' a^{-1})$$

and we get the identification

$$\mu^{-1}(\mathcal{O}) = \text{Im } \hat{\mathbf{b}}_{(e,0)}$$

Analogously, we define an equivariant map $\tilde{\zeta} : LD \rightarrow L(G^* \times \mathfrak{g})$ such that $\tilde{\zeta}(l \cdot [m]) = \hat{\mathbf{b}}(l, \tilde{\zeta}([m]))$ and get a factorization of $\hat{\Phi}$ through T^*LG^* as $\hat{\Phi} = \tilde{\mu} \circ \tilde{\zeta}_{\tilde{h}_o}$, by symplectic

maps, depicted in the diagram

$$\begin{array}{ccc}
 (\mathcal{O}; \omega_{KK}) & \xleftarrow{\tilde{\mu}} & (\tilde{\mu}^{-1}(\mathcal{O}); \tilde{\omega}_o|_r) \\
 \hat{\Phi} \uparrow & & \nearrow \tilde{\zeta}_{\mathfrak{h}_o} \\
 (\Omega D; \omega_{\Omega D}) & &
 \end{array} \tag{19}$$

for each $\tilde{h}_o \in G^*$.

III- POISSON-LIE T-DUALITY

Gluing together diagrams (15) and its mirror image, we recover the commutative four vertex diagram (1) relating the phase spaces LT^*G and LT^*G^* , belonging to σ -models with dual targets G and G^* , through vertex $(\Omega D, \omega_{\Omega D})$ and $(L\mathfrak{d}_\Gamma^*, \{, \}_{KK})$ and with arrows being LD^\wedge -equivariant (pre)symplectic maps.

Actually, the relation holds provided there exist a non trivial intersection region of the images of the momentum maps μ and $\tilde{\mu}$ in $(L\mathfrak{d}_\Gamma)^*$, that means, if there exist a set of points $(g, \eta) \in L(G \times \mathfrak{g}^*)$ and $(\tilde{h}, Z) \in L(G^* \times \mathfrak{g})$ satisfying (see eqs. (11) and (18)) the identity

$$(\psi(\eta), 1) = \widehat{Ad}_{\tilde{h}^{-1}g}^{LD^*}(\psi(Z), 1)$$

Because μ and $\tilde{\mu}$ are equivariant momentum maps, the intersection region extends to the whole coadjoint orbit of the point $\mu(g, \eta) = \tilde{\mu}(\tilde{h}, Z)$ in $(L\mathfrak{d}_\Gamma)^*$, establishing a connection between LD^\wedge -orbits in LT^*G and LT^*G^* . It can be seen that this common region coincides with the pure central extension orbit

$$\mathcal{O} = \text{Im } \mu \cap \text{Im } \tilde{\mu}$$

and it is a isomorphic image of the WZW reduced space, so that we may refine diagram (1) to get a connection between the phase spaces of sigma models on dual targets and WZW

model on the associated Drinfeld double group

$$\begin{array}{ccccc}
(\mu^{-1}(\mathcal{O}); \omega_o|_{\mu^{-1}}) & \xrightarrow{\mu} & (\mathcal{O}; \omega_{KK}) & \xleftarrow{\tilde{\mu}} & (\tilde{\mu}^{-1}(\mathcal{O}); \tilde{\omega}_o|_{\tilde{\mu}^{-1}}) \\
& & \downarrow \hat{\Phi}^{-1} & & \\
& \swarrow \varsigma_{g_o} & (\Omega D; \omega_{\Omega D}) & \searrow \tilde{\varsigma}_{\tilde{h}_o} & \\
\end{array} \tag{20}$$

Poisson-Lie T-duality is then accurately characterized by restricting this diagram to the symplectic leaves in $\mu^{-1}(\mathcal{O})$ and $\tilde{\mu}^{-1}(\mathcal{O})$. In fact, let us denote by $S(g_o) \subset \mu^{-1}(\mathcal{O})$ and $\tilde{S}(\tilde{h}_o) \subset \tilde{\mu}^{-1}(\mathcal{O})$ the symplectic leaves defined by the maps $\varsigma_{g_o} : \Omega D \rightarrow \mu^{-1}(\mathcal{O})$ and $\tilde{\varsigma}_{\tilde{h}_o} : \Omega D \rightarrow \tilde{\mu}^{-1}(\mathcal{O})$, respectively, then the composition of arrows

$$\begin{array}{ccccc}
S(g_o) & \xrightarrow{\mu} & \mathcal{O} & \xleftarrow{\tilde{\mu}} & \tilde{S}(\tilde{h}_o) \\
& & \downarrow \hat{\Phi}^{-1} & & \\
& & \Omega D & \nearrow \tilde{\varsigma}_{\tilde{h}_o} & \\
\end{array} \tag{21}$$

defines the duality *T-duality* transformation

$$\begin{aligned}
\Psi_{\tilde{h}_o} : S(g_o) &\longrightarrow S^*(\tilde{h}_o) \\
\Psi_{\tilde{h}_o}(g, \lambda) &= \left(\tilde{b}(\tilde{h}_o)_a, (a_{\tilde{h}_o})' (a_{\tilde{h}_o})^{-1} \right)
\end{aligned} \tag{22}$$

where $[a\tilde{b}] \in \Omega D$ is a based loop such that $(g, \lambda) = \hat{\mathbf{d}}([a\tilde{b}], (g_o, 0))$. This are nothing but the duality transformations given in [1] and [10]. Obviously, as a composition of symplectic maps, $\Psi_{\tilde{h}_o}$ is a canonical transformation, and a hamiltonian vector field tangent to $S(g_o)$ is mapped onto a hamiltonian vector field tangent to $\tilde{S}(\tilde{h}_o)$. Passing through ΩD by $\hat{\Phi}^{-1}$, as showed in the diagram, allows to switch ΩD to the opposite factorization $[a\tilde{b}]_B \rightarrow [\tilde{b}_a a_{\tilde{b}}]_B$ before to reach $\tilde{S}(\tilde{h}_o)$ by applying

$$\tilde{\varsigma}_{\tilde{h}_o} : [a\tilde{b}]_B \longrightarrow [\tilde{b}_a a_{\tilde{b}}]_B \longrightarrow \hat{\mathbf{b}}([\tilde{b}_a a_{\tilde{b}}]_b, (\tilde{h}_o, 0))$$

for $(\tilde{h}_o, 0) \in \tilde{\mu}^{-1}(0, 1)$.

Observe that diagram (1) can also be constructed for an arbitrary bicrossed product $D = G \bowtie M$ with a Lie algebra $\mathfrak{g} \bowtie \mathfrak{m}$ supplied with a non degenerate, symmetric, invariant bilinear form, replacing the vertex LT^*G by $L(G \times \mathfrak{m})$. Now this vertex carries a presymplectic structure defined by the pullback $\iota_G^* \omega_\Gamma^R$ and

$$\begin{aligned} L(G \times \mathfrak{m}) &\xrightarrow{\iota_G} LD \times L\mathfrak{d}^* \\ (g, \alpha) &\longmapsto (g, Ad_{g^{-1}}^{D*} \psi(\alpha) + C_k(g)) \end{aligned}$$

recovering the generalization of PL-T duality introduced in [13].

IV- COLLECTIVE HAMILTONIANS AND DUALITY TRANSFORMATIONS

Using the *geometrical* or *kinematical* information of the diagram (21) we now address to impose the appropriate dynamics in order it can be mapped through the arrows giving dynamical T-duality transformations.

To this end , we observe that $\Omega D = \Phi^{-1}(\mathcal{O})$ and the symplectic leaves in $\mu^{-1}(\mathcal{O})$, $\tilde{\mu}^{-1}(\mathcal{O})$ are replicas of the coadjoint orbit \mathcal{O} and, because of their equivariance, their tangent bundles are locally isomorphic to that of \mathcal{O} . As \mathcal{O} is in the vertex linking the three models, it is clear that T-duality transformation (21) exist at the level of hamiltonian vector fields for each hamiltonian vector on \mathcal{O} . So, it enough to select a hamiltonian vector field in \mathcal{O} and symplectic leaves in $\mu^{-1}(\mathcal{O})$ and $\tilde{\mu}^{-1}(\mathcal{O})$ to obtain a couple of T-dual related hamiltonian vector fields and, whenever they exist, a couple T-dual related solution curves belonging to some kind of dual sigma models.

In terms of hamiltonian functions, once a suitable function $\mathfrak{h} : (L\mathfrak{d}_\Gamma)^* \longrightarrow \mathbb{R}$ is fixed we have the corresponding hamiltonian function on $\mu^{-1}(\mathcal{O})$ and $\tilde{\mu}^{-1}(\mathcal{O})$ by pulling-back it through the momentum maps μ and $\tilde{\mu}$, so that the hamiltonian function restricted to $\mu^{-1}(\mathcal{O})$, $\tilde{\mu}^{-1}(\mathcal{O})$ and ΩD are in the so called *collective Hamiltonian form* [21]: $\mathfrak{h} \circ \mu$, $\mathfrak{h} \circ \tilde{\mu}$ and $\mathfrak{h} \circ \Phi$.

This ensures that the corresponding Hamiltonian vector fields will be tangent to the LD orbits. Moreover, a Hamiltonian vector field in $\widehat{Ad}_{l^{-1}}^{LD*}(0,1) \in \mathcal{O}$ is of the form $\widehat{ad}_{\mathcal{L}_h}^{LD*} \widehat{Ad}_{l^{-1}}^{LD*}(0,1)$ where $\mathcal{L}_h : (L\mathfrak{d}_\Gamma)^* \longrightarrow L\mathfrak{d}_\Gamma$ is the corresponding Legendre transformation, and the solution curves are determined from the solution of the differential equation

on LD

$$d_t l \, l^{-1} = \mathcal{L}_{\mathbf{h}}(\gamma(t)) \quad (23)$$

where $\gamma(t)$ is the trajectory of the hamiltonian vector field corresponding to \mathbf{h} . In fact, from the curve $l(t) \subset LD$, with $l(t=0) = e$, solution of the equation so that $\gamma(t) = \widehat{Ad}_{l(t)}^{LD*} \gamma(0)$, the solutions for the collective hamiltonian vector fields on T^*LG , ΩD and T^*LG^* are

$$\begin{cases} \hat{\mathbf{d}}(l(t), (g_0, \eta_0)) \\ [l(t)l_0] \\ \hat{\mathbf{b}}(l(t), (\tilde{h}_0, Z_0)) \end{cases}$$

respectively, with $l(t)$ given by (23) and for $(g_o, \eta_o) \in \mu^{-1}(\gamma(0))$, $[l_0] \in \hat{\Phi}^{-1}(\gamma(0))$, $(\tilde{h}_0, Z_0) \in \tilde{\mu}^{-1}(\gamma(0))$.

We see that duality transformations between T^*LG and T^*LG^* involve finding the curve $l(t)$ solution to eq. (23) and using the two factorizations of the double $D = G \bowtie G^* \sim G^* \bowtie G$. The generating functional on the dualizable subspaces is given in terms of the potentials ϑ_o and $\tilde{\vartheta}_o$ of the symplectic forms on the dual phase-spaces by [9]

$$\begin{aligned} d_V F[g, \tilde{g}] &= \vartheta_o - \tilde{\vartheta}_o \Big|_{S_R \times \tilde{S}_R} \\ &= \left\langle g^{-1} dg, \tilde{h} \tilde{h}^{-1} \right\rangle - \left\langle \tilde{g}^{-1} d\tilde{g}, h h^{-1} \right\rangle \\ &= - \int_{S^1} l^* (\iota_V \omega^{STS}) \end{aligned}$$

where V is a vector field along the loop $l = g\tilde{h} = \tilde{g}h$ in D and ω^{STS} is the symplectic form on the double D [19], [8]. This leads to the well known generating functional formula [1] for PLT-duality.

Also note that, in order to have a non trivial duality, restriction to the common sector in $(L\mathfrak{d}_\Gamma)^*$ where all the moment maps intersect is required, i.e., to the coadjoint orbit \mathcal{O} . This is why the study of the pre-images $\mu^{-1}(\mathcal{O})$ and $\tilde{\mu}^{-1}(\mathcal{O})$ of the last section becomes relevant. So, from now on, we shall refer to this pre-images as *dualizable* or *admissible* subspaces.

Now, a couple of remarks are in order. First, note that an analogous diagram to (1) can be constructed by replacing one of the phase spaces by any LD^\wedge -hamiltonian space. The same statements will hold for collective hamiltonian dynamics and so we can construct the corresponding duality transformations. This will lead us, as special cases, to *Buscher's duality* introduced in [1] and to duality between different factorizations of the Drinfeld double bialgebra $\mathfrak{d} = m + m^*$, some giving the so called PLT-plurality [28].

We also like to remark, before passing to the next section, that even when the central extension of the loop group LD does not exist, the same diagrams can be constructed and all the statements about collective dynamics (and so all about T-duality and duality transformations) still hold after replacement of the dual of centrally extended loop algebra $(L\mathfrak{d}_\Gamma)^*$ by $L\mathfrak{d}_{Aff}^*$ with the affine Poisson structure defined by the cocycle Γ and the affine coadjoint action defined in section I.

In the following subsections, we shall study the dynamics of collective Hamiltonians and the corresponding lagrangian formulation for the T-dual WZW and sigma models.

Hamiltonian and Lagrangian WZW model

The WZW-model reduced space ΩD is mapped into the coadjoint orbit \mathcal{O} by the momentum map $\hat{\Phi} : \Omega D \longrightarrow \mathcal{O}$ associated to the residual left invariance (5). So, let us consider a Hamiltonian $H_{WZW}(g, \eta)$ for the chiral WZW model which, when restricted to the reduced space $\Omega D = \left(\hat{J}^R\right)^{-1}(0, 1)/D$, it is in collective form

$$H_{WZW}(l, \varphi)|_{\Omega D} = \mathfrak{h} \circ \hat{\Phi}([l])$$

for some function $\mathfrak{h} : (L\mathfrak{d}_\Gamma)^* \rightarrow \mathbb{R}$. We shall consider a quadratic Hamiltonian which, having in mind that $\hat{J}^R(l, \varphi) = (k\psi(l^{-1}l') - \varphi, 1)$ (2), can be written in general form as

$$H_{WZW}(l, \varphi) = \frac{k^2}{2} (l'l^{-1}, \mathbb{L}_1 l'l^{-1})_{L\mathfrak{d}} + \langle Ad_{l^{-1}}^{LD*} \varphi, \mathbb{L}_2 l'l^{-1} \rangle + \frac{1}{2} \langle Ad_{l^{-1}}^{LD*} \varphi, \mathbb{L}_3 \bar{\psi} (Ad_{l^{-1}}^{LD*} \varphi) \rangle$$

for some linear self adjoint operators $\mathbb{L}_i : \mathfrak{d} \longrightarrow \mathfrak{d}$. The equations of motion (23) for this Hamiltonian are

$$\begin{aligned} \dot{l}l^{-1} &= k \mathbb{L}_2 l'l^{-1} + \mathbb{L}_3 \bar{\psi} (Ad_{l^{-1}}^{LD*} \varphi) \\ \dot{\varphi} &= k \left(Ad_{l^{-1}}^{LD} \left(k (\mathbb{L}_1 + \mathbb{L}_2) (l'l^{-1}) + (\mathbb{L}_2 + \mathbb{L}_3) \bar{\psi} (Ad_{l^{-1}}^{LD*} \varphi) \right) \right)' \\ &\quad - k \text{ad}_{l^{-1}l'}^{L\mathfrak{d}^*} \psi \left(Ad_{l^{-1}}^{LD} \left(k \mathbb{L}_2 l'l^{-1} + \mathbb{L}_3 \bar{\psi} (Ad_{l^{-1}}^{LD*} \varphi) \right) \right) \end{aligned} \tag{24}$$

When restricted to $\left(\hat{J}^R\right)^{-1}(0, 1) = \{(l, k\psi(l^{-1}l')) / l \in LD\} \cong LD$, they become into

$$\begin{aligned} \dot{l}l^{-1} &= k (\mathbb{L}_2 + \mathbb{L}_3) l'l^{-1} \\ \frac{d}{dt} \psi(l^{-1}l') &= k \left(Ad_{l^{-1}} \left((\mathbb{L}_1 + \mathbb{L}_2) (l'l^{-1}) + (\mathbb{L}_2 + \mathbb{L}_3) (l'l^{-1}) \right) \right)' \\ &\quad - k \text{ad}_{l^{-1}l'}^{L\mathfrak{d}^*} Ad_l^{LD*} \psi \left((\mathbb{L}_2 + \mathbb{L}_3) l'l^{-1} \right) \end{aligned}$$

Observe that for $\mathbb{L}_1 = -\mathbb{L}_2$, the second equation is derived from the first one, ensuring the Hamiltonian vector fields are tangent to the reduced submanifold $\left(\hat{J}^R\right)^{-1}(0,1)$. Thus, a suitable quadratic Hamiltonian for the WZW-model must have the form

$$H_{wzw}(l, \varphi) = k \left(\bar{\psi} \left(Ad_{l^{-1}}^{LD*} \varphi \right) - \frac{k}{2} l' l^{-1}, \mathbb{L}_2 l' l^{-1} \right)_{L\mathfrak{d}} + \frac{1}{2} \left(\bar{\psi} \left(Ad_{l^{-1}}^{LD*} \varphi \right), \mathbb{L}_3 \bar{\psi} \left(Ad_{l^{-1}}^{LD*} \varphi \right) \right)_{L\mathfrak{d}} \quad (25)$$

that, when reduced to ΩD , becomes into

$$H_{wzw}(l, \varphi)|_{\Omega D} = \frac{1}{2} \left(\bar{\psi} \left(\hat{\Phi}([l]) \right), (\mathbb{L}_2 + \mathbb{L}_3) \bar{\psi} \left(\hat{\Phi}([l]) \right) \right)_{L\mathfrak{d}} \quad (26)$$

unveiling its collective form in the momentum map $\hat{\Phi}$, for a quadratic Hamiltonian function $\mathbf{h} : (L\mathfrak{d}_\Gamma)^* \rightarrow \mathbb{R}$.

In order to pass to ΩD , we observe that $l = l_o \in D$ and the reduced equation of motion implies $\dot{l}_o l_o^{-1} = 0$, so that

$$\frac{d[l]}{dt} [l]^{-1} = k \mathbb{L} \frac{d[l]}{d\sigma} [l]^{-1}$$

with $\mathbb{L} = \mathbb{L}_2 + \mathbb{L}_3$, which is derived from (23). Finally, it is easy to see that the corresponding action functional is

$$S_{wzw}(l) = \frac{1}{2} \int_{\Sigma} \langle \partial_{\theta} l l^{-1}, \partial_t l l^{-1} \rangle + \frac{1}{12} \int_B \langle d l l^{-1}, [d l l^{-1}, d l l^{-1}] \rangle + \frac{1}{2} \int_{\Sigma} \langle \partial_{\theta} l l^{-1}, \mathbb{L} \partial_{\theta} l l^{-1} \rangle \quad (27)$$

where Σ is a 1+1 domain with a periodic variable θ , and B is a 3 dimensional domain such that $\partial B = \Sigma$. The "initial values" for the Hamiltonian equation of motion (24) which comes from (23) are boundary conditions for the fields $l(\sigma, t)$. This boundary conditions fix the topology of Σ , the main examples are: if the condition is to be defined for all non negative time and $l(\sigma, t = 0) = e$ for all σ , then the domain of the fields Σ has the topology of the disc; if the condition is to be defined for all finite time and $l(\sigma, t = 0) = l_0(\sigma)$ for some $l_0 \in LD$, then the domain of the fields Σ has the topology of the cylinder.

Now, the first two terms on the action give the potential 1-form for the symplectic 2-form $\omega_{\Omega D}$ in ΩD and the third term is the corresponding hamiltonian. We recognize here the WZW model first proposed by Klimcik and Severa if we take a specific choice of the operator \mathbb{L} . Up to the moment, there are no constrains on this operators but we shall see below that this constrains appear in order to reproduce sigma model like lagrangians on the targets G and G^* and we also describe how the boundary conditions on the fields get mapped to the dualizable subspaces.

Hamiltonian and Lagrangian T-dual sigma models

As explained above, classical T -duality is a consequence of a common collective dynamics on the non trivial intersection of the images of momentum maps of systems whose phase spaces are LD^\wedge -modules. This dynamics is fixed by a Hamiltonian function $h : (L\mathfrak{d}_\Gamma)^* \longrightarrow \mathbb{R}$ and the equation of motion (23) describes de hamiltonian vector fields mapped by the momentum maps. Analyzing the dynamics of the WZW-model in the previous section, we fixed the collective dynamics to be a quadratic Hamiltonian function so that the hamiltonian function on the sigma model phase space T^*LG get fixed to be

$$H_\sigma(g, \eta) = \frac{1}{2} (\bar{\psi}(\mu(g, \eta)), \mathbb{L}\bar{\psi}(\mu(g, \eta)))_{L\mathfrak{d}}$$

We note that only the symmetric part of the operator $(\mathbb{L}_2 + \mathbb{L}_3)$ with respect to the bilinear form $(,)_{L\mathfrak{d}}$ contributes. So we call this symmetric part $\mathcal{E} : \mathfrak{d} \rightarrow \mathfrak{d}$ and, using eq. (18), we write

$$H_\sigma(g, \eta) = \frac{1}{2} (Ad_g^{LD}\eta + k g'g^{-1}, \mathcal{E}(Ad_g^{LD}\eta + k g'g^{-1}))_{L\mathfrak{d}} \quad (28)$$

In order to recover the Lagrangian functional associated to this hamiltonian, we infer the inverse Legendre transformation from the Hamilton equation of motion for g

$$g^{-1}\dot{g} = \Pi_{\mathfrak{g}}\mathcal{E}_g\Pi_{\mathfrak{g}^*}(\eta) + \Pi_{\mathfrak{g}}\mathcal{E}_g\Pi_{\mathfrak{g}}(g^{-1}g')$$

where $\mathcal{E}_g = Ad_{g^{-1}}^{LD}\mathcal{E}Ad_g^{LD}$. For simplicity, we set from now on $k = 1$ and assume the Legendre transformation is non-singular which require the operator $\Pi_{\mathfrak{g}}\mathcal{E}_g\Pi_{\mathfrak{g}^*} : \mathfrak{g}^* \longrightarrow \mathfrak{g}$ to be invertible for all $g \in G$. However, it must be remarked that non invertibility would give rise to constraints and gauge symmetries, leading to coset sigma models [29] and constrained systems like the WZNW [12] for an appropriate choice of the kernel.

Let us name $\mathcal{G}_g = (\Pi_{\mathfrak{g}}\mathcal{E}_g\Pi_{\mathfrak{g}^*})^{-1} : \mathfrak{g} \longrightarrow \mathfrak{g}^*$ and $\mathcal{B}_g = -\mathcal{G}_g \circ \Pi_{\mathfrak{g}}\mathcal{E}_g\Pi_{\mathfrak{g}} : \mathfrak{g} \longrightarrow \mathfrak{g}^*$, then

$$\eta = \mathcal{G}_g(g^{-1}\dot{g}) + \mathcal{B}_g(g^{-1}g')$$

Now a question arise: when does the resulting Lagrangian is in a sigma model form? Recall that the Lagrangian of a (non singular) sigma model can be write in the form [9]

$$L = \langle g^{-1}g_-, (\mathbf{G}_g + \mathbf{B}_g)g^{-1}g_+ \rangle \quad (29)$$

for \mathbf{G}_g being a symmetric invertible operator (the metric), \mathbf{B}_g an antisymmetric operator (the B -field), both from $\mathfrak{g} \longrightarrow \mathfrak{g}^*$ and depending on the point $g \in G$ (the symmetry properties referred to the bilinear form given by the pairing \langle, \rangle).

The answer to this question is given by the following Lemma, in terms of the algebraic properties of the operator \mathcal{E}_g in the vector space $\mathfrak{g} \oplus \mathfrak{g}^*$

Lemma: *The Lagrangian coming from the collective hamiltonian given by the symmetric operator \mathcal{E} defines a sigma model given by the Lagrangian(29) with $\mathbf{G}_g = \mathcal{G}_g$ and $\mathbf{B}_g = \mathcal{B}_g$ iff one of the following equivalent conditions are fulfilled for each G*

1. \mathcal{B}_g is antisymmetric and $\mathcal{G}_g - \mathcal{B}_g(\mathcal{G}_g)^{-1}\mathcal{B}_g = \Pi_{\mathfrak{g}^*}\mathcal{E}_g\Pi_{\mathfrak{g}}$
2. $(\mathcal{E}_g)^2 = 1$
3. $\mathcal{E}^2 = 1$
4. As a block matrix in $\mathfrak{g} \oplus \mathfrak{g}^*$, we have

$$\mathcal{E}_g = \begin{pmatrix} -(\mathcal{G}_g)^{-1}\mathcal{B}_g & (\mathcal{G}_g)^{-1} \\ \mathcal{G}_g - \mathcal{B}_g(\mathcal{G}_g)^{-1}\mathcal{B}_g & \mathcal{B}_g(\mathcal{G}_g)^{-1} \end{pmatrix} \quad (30)$$

5. $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}^* = \mathcal{E}_g^+ \oplus \mathcal{E}_g^-$ where \mathcal{E}_g^\pm are the ± 1 eigenspaces of \mathcal{E}_g having the dimension equal to $\dim \mathfrak{g}$ and being orthogonal to each other.

Conversely, the Hamiltonian coming from a Lagrangian (29) is in collective motion form for the moment map μ and quadratic non singular hamiltonian on $L\mathfrak{d}$ if the operator defined by (30) satisfies $\mathcal{E}_g = Ad_{g^{-1}}^{LD}\mathcal{E}_eAd_g^{LD}$.

The proof of this lemma is straightforward. It gives the exact relation between the collective hamiltonian form and the sigma model data [1][30]. We remark that the equivalences rely only on the algebraic properties of the vector space $\mathfrak{g} \oplus \mathfrak{g}^*$ with the pairing as a bilinear form and the symmetry of \mathcal{E} (\mathcal{E}_g will be also symmetric by the Ad - D invariance of \langle, \rangle).

Note that this kind of operators can be given by (generalized) complex structures on the double algebra. This observation becomes relevant in the supersymmetric case [25].

In order to give a more explicit expression for the sigma model Lagrangian, we introduce graph coordinates for the eigenspaces \mathcal{E}_g^\pm on $\mathfrak{g} \oplus \mathfrak{g}^*$ (following the description of [13])

$$\mathcal{E}_g^\pm = \{X \oplus (\mathcal{B}_g X \pm \mathcal{G}_g X), X \in \mathfrak{g}\}$$

This can be easily inferred from the matrix form of the operator \mathcal{E}_g . Now, using the dual graph coordinates

$$\mathcal{E}_g^\pm = \{\phi \oplus (\mathcal{B}_g \pm \mathcal{G}_g)^{-1}\phi, \phi \in \mathfrak{g}^*\}$$

and relating them to the ones for $g = e$, noting that $v \in \mathcal{E}_g^\pm$ iff $Ad_g v \in \mathcal{E}_e^\pm$ so

$$\begin{aligned} Ad_{g^{-1}}(\phi \oplus (\mathcal{B}_e \pm \mathcal{G}_e)^{-1}\phi) &= \Pi_{\mathfrak{g}^*} Ad_{g^{-1}}\phi \oplus Ad_{g^{-1}}((\mathcal{B}_e \pm \mathcal{G}_e)^{-1}\phi + Ad_g \Pi_{\mathfrak{g}} Ad_{g^{-1}}\phi) \\ &= \Pi_{\mathfrak{g}^*} Ad_{g^{-1}}\phi \oplus (\mathcal{B}_g \pm \mathcal{G}_g)^{-1}(\Pi_{\mathfrak{g}^*} Ad_{g^{-1}}\phi) \end{aligned}$$

and we can deduce that

$$(\mathcal{B}_g \pm \mathcal{G}_g)^{-1}(\phi) = \Pi_{\mathfrak{g}} Ad_{g^{-1}} \Pi_{\mathfrak{g}} ((\mathcal{B}_e \pm \mathcal{G}_e)^{-1}\phi + Ad_g \Pi_{\mathfrak{g}} Ad_{g^{-1}} \Pi_{\mathfrak{g}^*} Ad_g \phi)$$

Finally

$$\Pi_{\mathfrak{g}} Ad_g \Pi_{\mathfrak{g}} (\mathcal{B}_g \pm \mathcal{G}_g)^{-1} \Pi_{\mathfrak{g}^*} Ad_g \Pi_{\mathfrak{g}^*} = (\mathcal{B}_e \pm \mathcal{G}_e)^{-1} + \pi(g) \quad (31)$$

where $\pi(g) = \Pi_{\mathfrak{g}} Ad_g \Pi_{\mathfrak{g}^*} Ad_{g^{-1}} \Pi_{\mathfrak{g}^*} = -\pi^R(g^{-1})$, and π^R gives the Poisson bivector right translated to the origin on the Poisson-Lie group G coming from the Lie-bialgebra structure of $(\mathfrak{g}, \mathfrak{g}^*)$ (see [3], for example).

So, coming back to the sigma model Lagrangian, we have

$$L = \langle g^{-1}g_-, (\mathcal{B}_g + \mathcal{G}_g)g^{-1}g_+ \rangle = \langle g_-g^{-1}, ((\mathcal{B}_e + \mathcal{G}_e)^{-1} + \pi(g))^{-1}g_+g^{-1} \rangle$$

where in the last expression we recognize the Lagrangian of the sigma model on the target G first introduced by Klimcik and Severa [1].

The corresponding dual construction can be repeated following analogous steps, interchanging the roles of G and G^* , yielding the dual sigma model Lagrangian on the target G^*

$$\tilde{L} = \langle \tilde{g}_- \tilde{g}^{-1}, ((\mathcal{B}_e + \mathcal{G}_e) + \tilde{\pi}(\tilde{g}))^{-1} \tilde{g}_+ \tilde{g}^{-1} \rangle$$

where $\tilde{\pi}$ is the corresponding Poisson bivector of the Poisson-Lie structure on G^* , coming from the bialgebra $(\mathfrak{g}^*, \mathfrak{g})$.

From the construction developed on the preceding sections we know that this two models are "dual" to each other and to the WZW model defined by (27), in the sense that solutions contained in the dualizable subspace in one model can be mapped through the coadjoint orbit \mathcal{O} to the other model, and the *duality transformation* involves finding the appropriate curve in LD^\wedge and generating the dual flows by the action of this curve on the initial value.

Conversely, we can ask when a generic sigma model on the target G will be dualizable in the above sense. The answer to this question within the Lagrangian formalism was given in the pioneer works [1]. So we conclude this section giving the exact relation between the Lagrangian dualizability conditions (the so called Poisson-Lie symmetry of the sigma model Lagrangian) and the information contained in our Hamiltonian approach. To that end, following [1], we introduce the following 1-form over Σ with values in \mathfrak{g}^*

$$J = \Pi_{\mathfrak{g}^*}(G + B)g^{-1}g_+dx^+ - \Pi_{\mathfrak{g}^*}(G - B)g^{-1}g_-dx^-$$

and we recall that a sigma model given by (29) is called (right) PL-symmetric with respect to \mathfrak{g}^* if

$$J = \frac{1}{2}[J, J]_{\mathfrak{g}^*} \quad (32)$$

over the solutions and where a Lie bracket is given on \mathfrak{g}^* . It was shown that this equations require certain compatibility conditions, namely, the bracket on \mathfrak{g}^* should be such that $(\mathfrak{g}, \mathfrak{g}^*)$ becomes a Lie bialgebra, and that the G -dependent operators \mathbf{G}_g and \mathbf{B}_g defining the sigma model should satisfy the compatibility condition

$$\mathcal{L}_{X^{L(g)}} \langle Y, (G - B)Z \rangle = - \left\langle X, ad_{\Pi_{\mathfrak{g}^*}(G-B)Z}^{\mathfrak{g}^*}(\Pi_{\mathfrak{g}^*}(G + B)Y) \right\rangle \quad (33)$$

for $X, Y, Z \in \mathfrak{g}$, $\mathcal{L}_{X^{L(g)}}$ is the Lie derivative with respect to X^L , the left invariant vector field on LG generated by X for all $g \in LG$. Note that if $\mathbf{G}_g - \mathbf{B}_g$ is G -independent, this compatibility condition on \mathbf{G}_g and \mathbf{B}_g defines a quasitriangular structure on the Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$ [13] [30]..

This PL-symmetry condition defines, when Σ is contractile (for example, with the topology of the disc), a function $\tilde{h} : \Sigma \rightarrow G^{ast}$ such that $J = d\tilde{h}\tilde{h}^{-1}$ and is easy to see that the equation (32) becomes equivalent to

$$(1 \pm \mathcal{E})l_{\mp}l^{-1} = 0$$

for $l = g\tilde{h}$ and $\mathcal{E} = Ad_g\mathcal{E}_gAd_{g^{-1}}$ with \mathcal{E}_g the operator given by the matrix (30) in terms of \mathbf{G}_g and \mathbf{B}_g . Moreover, the compatibility condition first order equation (33) defines how \mathbf{G}_g and \mathbf{B}_g depend on $g \in G$ and it can be proved that it is equivalent to the fact that the operator \mathcal{E} just defined is constant for all $g \in G$, so \mathcal{E}_e plays the role of initial values for these equations. We see that \mathcal{E} fulfills the conditions of the previous Lemma, so we have

Lemma: *Let Σ be contractible and with a periodic spatial coordinate σ . A sigma model Lagrangian given in the form (29) is (right) PL symmetric respect to g^* iff the corresponding hamiltonian function on LT^*G is in collective motion form for the moment map μ and the quadratic function $\langle v, \mathcal{E}v \rangle$ on $L\mathfrak{d}_\Gamma^*$ defined by a symmetric and idempotent operator \mathcal{E} on \mathfrak{d} .*

In the case of the cylinder topology (remember the relation we stated between the topology of the 1+1 domain and the initial values for the Hamiltonian equations of motion for $l(\sigma, t)$), this is also true once we have imposed a unit monodromy constraint for the current J (see below).

Finally, we shall comment on the restriction to the dualizable subspaces. Up to now, we know by construction that there is a (unique up to constant G^* elements) correspondence between solutions of the dualizable sigma model (g, \tilde{h}) and hamiltonian integral curves $(g, \tilde{h}'\tilde{h}^{-1})$. Now, the image of such a solution through the momentum map μ lies in the coadjoint orbit $\mathcal{O}(\tilde{h}'\tilde{h}^{-1}, 1)$ inside $(L\mathfrak{d}_\Gamma)^*$ and it is easy to see that $\mathcal{O}(\tilde{h}'\tilde{h}^{-1}, 1) = \mathcal{O}$ iff \tilde{h} is periodic in the spatial variable (i.e., iff it is a loop for all t). So the restriction to the dualizable subspace can be expressed as a unit monodromy constrain on $J = d\tilde{h}\tilde{h}^{-1}$,

$$\tilde{h}(0, t) = \tilde{h}(2\pi, t)$$

or equivalently

$$P \int_\gamma d\tilde{h}\tilde{h}^{-1} = \tilde{e}$$

for any closed path γ homotopic to a constant time loop in Σ , and the same holds for the dual model, as first noted by Klimcik and Severa.

So, as the Lagrangian on the G target describes the Hamiltonian dynamics in the whole phase space LT^*G it is natural to ask what kind of models arises when we replace LT^*G^* by other phase-space such that it has a non-trivial intersection with the other coadjoint orbits (the image of the subspaces restricted by non-trivial monodromy conditions) $\mathcal{O}(\alpha, 1)$, with $\alpha \in L\mathfrak{g}^*$, which also lie in $\mu(LT^*G)$. Such models should have phase spaces consisting of non-closed paths in G^* (because of the non-trivial monodromy of α). Examples of this models are the ones given in [27].

The reader might also note that the Poisson structures on such open path spaces are closely related to the ones associated to the chirally extended WZW phase space [26], which

are also LD spaces, and from this point of view one could have a better understanding of the appearance of (finite dimensional) P-L symmetries generated by the monodromy matrix of the resulting open strings variables [36].

V- EXAMPLES

We will now give some examples to illustrate on the construction of the duality transformations and admissible subspaces for special simple choices of the double group and the Hamiltonian dynamics, recovering a full explicit description of known results on target space duality.

Abelian duality and $R \longrightarrow \frac{1}{R}$ momentum-winding interchange.

In this example, we take a trivial Lie bialgebra $(\mathfrak{g}, [,] = 0, \delta = 0)$, which has a trivial dual bialgebra and a trivial double $(\mathfrak{g}^*, [,] = 0, \delta = 0)$, $(\mathfrak{d}, [,] = 0, \delta = 0)$ respectively. Moreover, we take \mathfrak{g} to be the Lie algebra of the 1-dimensional abelian group $G = U_R(1)$ thought as a circle of radius R with group law the translation along the circle. Being the bilinear form $(\cdot, \cdot)_{\mathfrak{d}}$ on the double Lie algebra the pairing between \mathfrak{g} and its dual \mathfrak{g}^* , we can choose the dual group to be $G^* = U_{\frac{1}{R}}(1)$, the *dual* circle of radius $\frac{1}{R}$, since we can naturally think of \mathfrak{g} and its dual \mathfrak{g}^* as the corresponding tangent spaces at the origin and if we parametrize the group elements as Rx and $R^{-1}x$ with $x \in [0, 2\pi]$ then $\langle R\partial_x, R^{-1}d_x \rangle = 1$. We choose the (non-simply connected) double group to be $D = U_R(1) \times U_{R^{-1}}(1)$. The Poisson bracket on $L\mathfrak{d}^{\wedge*}$ is pure central extension $\{, \} \equiv \Gamma$, and we choose the Hamiltonian function on $L\mathfrak{d}^{\wedge*}$ to be

$$\mathfrak{H}(X, \xi, a) = \int_{S^1} d\sigma \left(\frac{1}{2R^2} \xi^2 + \frac{R^2}{2} X^2 \right)$$

The phase-spaces of the dual models are $LT^*U_R(1)$ and $LT^*U_{\frac{1}{R}}(1)$, with parametrized elements $\gamma = (\theta(\sigma), \pi(\sigma)) \in LT^*U_R(1)$ and $\tilde{\gamma} = (\tilde{\theta}(\sigma), \tilde{\pi}(\sigma)) \in LT^*U_{\frac{1}{R}}(1)$. The moment maps associated with the LD action are $\mu(\theta(\sigma), \pi(\sigma)) = (\frac{d}{d\sigma}\theta(\sigma) + \pi(\sigma), 1)$ and $\tilde{\mu}(\tilde{\theta}(\sigma), \tilde{\pi}(\sigma)) = (\frac{d}{d\sigma}\tilde{\theta}(\sigma) + \tilde{\pi}(\sigma), 1)$. The Hamiltonians on the phase-spaces written on the collective form are

$$H(\gamma; \sigma) = \int_{S^1} d\sigma \left[\frac{1}{2R^2} \pi(\sigma)^2 + \frac{R^2}{2} \left(\frac{d\theta}{d\sigma} \right)^2 \right]$$

$$\tilde{H}(\tilde{\gamma}; \sigma) = \int_{S^1} d\sigma \left[\frac{1}{2R^2} \left(\frac{d\tilde{\theta}}{d\sigma} \right)^2 + \frac{R^2}{2} \tilde{\pi}(\sigma)^2 \right]$$

The duality transformation $\Psi : \mu^{-1}(\mathcal{O}) \longrightarrow \tilde{\mu}^{-1}(\mathcal{O})$ can be constructed following the arrows of the diagram (1)

$$\begin{array}{ccc}
LT^*U_R(1) & & LT^*U_{1/R}(1) \\
(\theta(\sigma), \pi(\sigma)) \xrightarrow{\mu} (\theta'(\sigma), \pi(\sigma), 1) & \xleftarrow{\tilde{\mu}} & (\tilde{\theta}_o + \int_0^\sigma \pi(\zeta) d\zeta, \frac{d\theta}{d\sigma}(\sigma)) \\
\downarrow \hat{\Phi}^{-1} & \nearrow \zeta_{\tilde{\theta}_o} & \\
\theta(\sigma) - \theta(0) \times \int_0^\sigma d\zeta \pi(\zeta) & &
\end{array} \tag{34}$$

so we get

$$\Psi(\gamma(\sigma)) = \tilde{\gamma}(\sigma) = \left(\int_0^\sigma \pi(\zeta) d\zeta, \frac{d\theta}{d\sigma}(\sigma) \right)$$

Proceeding analogously, starting from the dual part, we obtain $\tilde{\Psi}$. By construction

$$\Psi^* \tilde{H} = H$$

The admissible subspace $\mu^{-1}(\mathcal{O})$ in $LT^*U_R(1)$ is of the form $\{(\theta, \tilde{\alpha}) \bullet (\theta_0, 0) = (\theta + \theta_0, \tilde{\alpha}') / : (\theta, \tilde{\alpha}) \in \Omega D\}$ since D is abelian and so the dressing actions are trivial. Similarly, $\tilde{\mu}^{-1}(\mathcal{O}) = \{(\tilde{\theta}, \alpha) \bullet (\tilde{\theta}_0, 0) = (\tilde{\theta} + \tilde{\theta}_0, \alpha') / : (\tilde{\theta}, \alpha) \in \Omega D\}$ in $LT^*U_{1/R}(1)$. We note that the elements in ΩD giving the duality transformations have $\theta(0) = 0$ or they have no *momentum zero modes* in their Fourier expansion. This corresponds, as in the general PL case, to the unit monodromy constraint (see [27]). Now, the topology of the $U(1)$ targets allows us to introduce a refined description of the dualizable subspaces.

As every element $(\theta, \tilde{\theta}) \in L(U_R(1) \times U_{1/R}(1))$ is classified by its homotopic class or "winding number" we then define the subsets

$$L(n, m) = \{(\theta, \tilde{\theta}) \in L(U_R(1) \times U_{1/R}(1)) / : \deg \theta = n \text{ and } \deg \tilde{\theta} = m\}$$

So $\mu^{-1}(\mathcal{O}) = \bigcup (L_R(n, m)) = \bigcup \{(\theta, \pi) / : \deg \theta = n \text{ and } \int_{S^1} \pi = 2\pi m\}$ and $\tilde{\mu}^{-1}(\mathcal{O}) = \bigcup (L_{1/R}(n, m)) = \bigcup \{(\tilde{\theta}, \tilde{\pi}) / : \deg \tilde{\theta} = n \text{ and } \int_{S^1} \tilde{\pi} = 2\pi m\}$. Moreover, it's easy to see that the Hamiltonian flows preserves these winding numbers and so the $L_R(n, m)$ and $L_{1/R}(n, m)$

are sub-Hamiltonian systems of $LT^*U_R(1)$ and $LT^*U_{\frac{1}{R}}(1)$ respectively, which lie inside the dualizable subspaces. Finally, we see that the duality transformation Ψ maps $L_R(n, m)$ to $L_{\frac{1}{R}}(m, n)$ interchanging $R \longrightarrow R^{-1}$ and the winding number n to be the momentum number in the dual model and the momentum number m to the winding number in the dual model. Hence we have recovered the momentum-winding duality transformation and the domain of these transformation in the phase spaces as described in [5] within our general framework.

Semi-abelian or non-abelian $G \longleftrightarrow g^*$ duality

In this example, we take the bialgebra \mathfrak{g} to be semi-trivial, that is, $(\mathfrak{g}, [\cdot, \cdot], \delta = 0)$. So $(\mathfrak{g}^*, [\cdot, \cdot] = 0, \delta)$ and the double $(\mathfrak{d}, [\cdot, \cdot], \delta)$ can be identified as a Lie algebra with the semi-direct product of $(\mathfrak{g}, [\cdot, \cdot])$ and $(\mathfrak{g}^*, [\cdot, \cdot] = 0)$ where \mathfrak{g} acts on \mathfrak{g}^* by the coadjoint representation. We will take G as a compact simple group and in role of G^* we consider \mathfrak{g}^* as an additive Abelian group, so the double group can be identified with the semidirect product $D = G \triangleright \mathfrak{g}^*$, with G acting by the coadjoint representation. The phase-spaces are LT^*G and $LT^*\mathfrak{g}^*$ and taking the product on $G \triangleright \mathfrak{g}^*$ to be

$$(g, \eta) \cdot (h, \lambda) = (gh, \lambda + Ad_h^{LG^*} \eta)$$

we have that $(g, \eta)^{-1} = (g^{-1}, -Ad_{g^{-1}}^{LG^*} \eta)$ and the momentum maps corresponding to the LD actions on them are $\mu(g, \eta) = (k \ g'g^{-1} + Ad_{g^{-1}}^{LG^*} \eta, 1)$ for $(g, \eta) \in L(G \times \mathfrak{g}^*)$, the left trivialization of LT^*G , and $\tilde{\mu}(\eta, X) = (X + \eta' + ad_X^{LG^*} \eta, 1)$ for $(\eta, X) \in LT^*\mathfrak{g}^*$. The coadjoint D and LD actions become

$$Ad_{(g, \eta)}^D(X, \xi) = (Ad_g^G X, Ad_{g^{-1}}^{G^*} \xi + Ad_{g^{-1}}^{G^*} ad_X^{G^*} \eta)$$

$$\widehat{Ad}_{(g, \eta)}^{LD^*}(0, 1) = ((g, \eta)' \cdot (g, \eta)^{-1}, 1) = (g'g^{-1} + Ad_{g^{-1}}^{LG^*} \eta', 1)$$

for $(g, \eta) \in LD$ acting on $L\mathfrak{d}^*$. The actions on the cotangent bundles can be derived from the dressing action for this particular case

$$\begin{aligned} (g, \eta) &= (g, 0) \cdot (e, \eta) = (e, \alpha) \cdot (g, 0) \implies g^\eta = g \quad , \quad \alpha^g = Ad_g^{LG^*} \alpha \\ (g, \eta) &= (e, Ad_{g^{-1}}^{LG^*} \eta) \cdot (g, 0) = (g, 0) \cdot (e, \eta) \implies \eta_g = Ad_{g^{-1}}^{LG^*} \eta \quad , \quad g_\eta = g \end{aligned}$$

so that

$$\begin{aligned}\hat{\mathbf{d}}((h, \xi), (g, \eta)) &= \hat{\mathbf{d}}((h, 0) \cdot (e, \xi), (g, \eta)) = (hg, \eta + (Ad_g^{LG*} \xi)') \\ \hat{\mathbf{b}}((h, \xi), (\eta, X)) &\equiv \hat{\mathbf{b}}((e, Ad_h^{LG*} \xi) \cdot (h, 0), (\eta, X)) = (Ad_h^{LG*} \xi + Ad_h^{LG*} \eta, Ad_h^G X + h'h^{-1})\end{aligned}$$

From here we can get the elements of $\mu^{-1}(\mathcal{O})$ and $\tilde{\mu}^{-1}(\mathcal{O})$

$$\mu^{-1}(\mathcal{O}) = \hat{\mathbf{d}}((h, \xi), (g_o, 0)) = (hg_o, Ad_{g_o}^{LG*} \xi')$$

$$\tilde{\mu}^{-1}(\mathcal{O}) = \hat{\mathbf{b}}((h, \xi), (\eta_o, 0)) = (Ad_h^{LG*} \xi + Ad_h^{LG*} \eta_o, h'h^{-1})$$

for $(h, \xi)(\sigma = 0) = (e, 0)$ and, following the arrows in the diagram (22),

$$\begin{array}{ccc} (hg_o, Ad_{g_o}^{LG*} \xi') & \xrightarrow{\mu} & \widehat{Ad}_{(h, \xi)}^{LD*}(0, 1) \xleftarrow{\tilde{\mu}} (Ad_h^{LG*}(\xi + \eta_o), h'h^{-1}) \\ & & \downarrow \hat{\Phi}^{-1} \nearrow \zeta_{\eta_o} \\ & & [(h, \xi)] \end{array}$$

we get the duality transformation

$$\Psi_{\eta_o} (hg_o, Ad_{g_o}^{LG*} \xi') = (Ad_h^{LG*}(\xi - \xi_o) + Ad_{(hh_o^{-1})^{-1}}^{LG*} \eta_o, h'h^{-1}) \quad (35)$$

The admissible subspace $\mu^{-1}(\mathcal{O})$ in LT^*G is

$$LD \cdot (e, 0) \sim (LG, \Omega \mathfrak{g}^*)$$

and $\tilde{\mu}^{-1}(\mathcal{O})$ in $LT^*\mathfrak{g}^*$ is

$$LD \cdot (0, 0) \sim (L\mathfrak{g}^*, \Omega G)$$

The inverse duality transformation can be computed following the arrows in (1) the other direction

$$\begin{array}{ccc} (hh_o^{-1}g_o, Ad_{g_o}^{LG*} \xi') & \xleftarrow{\mu} & \widehat{Ad}_{(h, \xi)}^{LD*}(0, 1) \xleftarrow{\tilde{\mu}} (Ad_h^{LG*}(\xi + \eta_o), h'h^{-1}) \\ & & \downarrow \hat{\Phi}^{-1} \nwarrow \zeta_{g_o} \\ & & [(h, \xi)] \end{array}$$

where $(\eta_o, 0) \in \tilde{\mu}^{-1}(\mathcal{O})$, so we get

$$\tilde{\Psi}_{g_o} (Ad_{h^{-1}}^{LG*} \xi + Ad_{h^{-1}}^{LG*} \eta_o, h'h^{-1}) = (hh_o^{-1} g_o, Ad_{g_o}^{LG*} \xi') \quad (36)$$

This duality transformations can be obtained from a generating functional

$$\Gamma(h, \xi) = - \int_{S^1} \langle \xi(\sigma), h'h^{-1} \rangle = - \int_{S^1} l^* \vartheta$$

as it is done in [5]. The last equality follows for $l = (h, \xi)$ from the general formula for the generating functional of the duality transformations and the fact that for $D = G \triangleright \mathfrak{g}^*$ the symplectic form on the Double is exact $\omega^{STS} = d\vartheta$. In ref. [5], they take a slightly different functional $\Gamma(\varphi^{-1}\varphi', \xi) = - \int_{S^1} \langle \xi(\sigma), \varphi^{-1}\varphi' \rangle$ that leads to equivalent duality transformations which can be derived within our framework by taking $LT^*G \sim L(G \times \mathfrak{g}^*)$ trivialized by right translations and the cocycle Γ on LD by $\Gamma(l) = l^{-1}l'$ and repeat the whole procedure of constructing the maps μ and ϕ in an analogous way changing left by right invariants. The discussion about the right domain for the duality transformations is, within our framework, very simple because by construction we know that the correct domains are given by the dualizable subspaces which we explicitly constructed and, in addition, we know that they will be invariant under any collective hamiltonian flow on the phase-spaces.

Conclusions

We have analyzed some relevant geometric properties of the loop spaces related to Poisson-Lie T-Duality, mainly centred on loop actions of the σ -models derived from the dressing transformation lifted to the cotangent bundle, with associated equivariant momentum maps. This allowed us to describe and understand many of the various aspects of this duality under the Hamiltonian formalism, like the explicit procedure of duality transformations, a precise identification of the dualizable subspaces and their relevance, and to reconstruct in a systematic way the well-known T-dual sigma model Lagrangians for suitable choices of the corresponding collective Hamiltonian dynamics. Moreover, this description allows to identify the relevant properties of the underlying information given by the models in order to be T-dual. In that way, we observed the same construction can reproduce known generalizations of PLTD as coset model dualities [29], duality for matched pairs [13], PLT-plurality [28] for different decompositions of the Drinfeld double, Buscher's duality [13][1]

and duality for monodromic strings models [27], because the underlying loop geometry enjoys the same properties for duality as the standard case. We believe that this approach can be generalized and adapted to cover many different (new) types of dualities becoming a useful geometrical approach for the study of T-dualities. For example, one can replace LT^*G^* by other Hamiltonian LD_Γ -space like the ones related to dual symmetric spaces [32] and, repeating the above construction, generate a new collection of T-dual models for each collective Hamiltonian choice. In addition, the construction itself will give the properties of the resulting models with respect to duality. This can also be applied for non-perfect doubles using the symplectic leaves decomposition of [8]. More generally, one can build up new types duality diagrams by considering symplectic groupoid or Poisson-Lie group actions instead of the usual hamiltonian ones. For finite dimensional PL actions, the construction given in this paper can be adapted to describe the duality observed between G and G^* Poisson-sigma models [31] with its corresponding boundary-bulk duality transformation. It would be also interesting to repeat the construction for the chirally extended WZNW spaces [26] which is both a Loop space and a finite dimensional PL space, and relate this actions to more general ones by the (Morita) equivalences of [34]. In the infinite dimensional PL case, we think that the relevant loop spaces for repeating the diagram construction should be closely related to the ones investigated in [24]. A more general approach based on groupoid actions might be used to study global properties of the dualities given in [9].

Finally, we hope that, as this is an explicit description of T-dualities in the Hamiltonian formalism, it will turn out to be useful to analyze the resulting quantum T-dualities under a quantization scheme adapted to the underlying geometry of the dual phase-spaces.

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